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The Borsuk–Ulam theorem for homotopy spherical space forms

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Abstract. In this work, we show for which odd-dimensional homotopy spherical space forms the Borsuk–Ulam theorem holds. These spaces are the quotient of a homotopy odd-dimensional sphere by a free action of a finite group. Also, the types of these spaces which admit a free involution are characterized. The case of even-dimensional homotopy spherical space forms is basically known.

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1. Introduction

The Borsuk–Ulam theorem [Bo] has been generalized in many directions. In [Gon] the author considered the case where the domain is a surface with a free involution and the target is $\mathbb{R}^2$. It follows from [Gon] that in some cases the result depends on the involution, in contrast with the classical case when the domain is the sphere. For this reason it is natural to consider the question about a Borsuk–Ulam-type theorem for triples $(X, \tau; Y)$, where X and Y are CW-complexes and τ is a free involution on X. *We say that the triple $(X, \tau; Y)$ satisfies the Borsuk–Ulam theorem if given any continuous map $f : X \to Y$ there exists a point $x \in X$ such that $f(x) = f(\tau(x))$.* In this work, we study the case where X is a $(2n-1)$-dimensional homotopy spherical space form and $Y = \mathbb{R}^{2n-1}$. Here a $(2n-1)$-*dimensional homotopy spherical space form*, denoted by $X(2n-1)$, is the quotient of a homotopy $(2n-1)$-sphere by a free cellular action of a finite group (see Section 3), where a *homotopy* $(2n-1)$-*sphere* is a $(2n-1)$-dimensional CW-complex which has the homotopy type of the $(2n-1)$-sphere. The case $n = 2$ is treated separately. The main results of the present work are the following.

Theorem 1.1. *A finite group H acts freely on some homotopy sphere Σ^{2n-1} in such a way that $X(2n-1) = \Sigma^{2n-1}/H$ admits a free involution τ if and only if H is one of the groups in the list in the third column of Table* 3 *(if $n > 2$) and of Table* 4 *(if $n = 2$).*

Theorem 1.2. *Let $X = X(3)$ be a three-dimensional spherical space form, and τ a free involution on X. The Sylow 2-subgroup of $\pi_1(X)$ is either*

(a) *trivial and generalized quaternion, or*
(b) *cyclic and nontrivial.*

In case (a) *the Borsuk–Ulam theorem holds for the triple $(X(2n-1), \tau; \mathbb{R}^{2n-1})$, while it does not hold in case* (b)*.*

Corollary 1.3. *The Borsuk–Ulam theorem holds for the triple $(X(3), \tau; \mathbb{R}^3)$ if and only if $\pi_1(X)$ is one of the following groups (of type* I, IIb, *and* IV, *respectively):*

(1) *$\mathbb{Z}/a$, with a odd;*
(2) *$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Q_{2^t})$, $t \geq 3$;*
(3) *$\mathbb{Z}/a \times T_i$, with a odd.*

Theorem 1.4. *Let $X = X(2n-1)$ be a $(2n-1)$-dimensional spherical space form, and τ a free involution on X, with $n > 2$. Then the Borsuk–Ulam theorem holds for the triple $(X(2n-1), \tau; \mathbb{R}^{2n-1})$ if and only if $\pi_1(X)$ has no element of order* 2*.*

Note that for $Y = \mathbb{R}^2$ the Borsuk–Ulam theorem always holds (see Remark 5.4). If X is an even-dimensional homotopy space form, then it is well known that $\pi_1(X)$ is either trivial or $\mathbb{Z}/2$. If $\pi_1(X)$ is trivial, then X is a homotopy sphere Σ^{2n}, and if $\pi_1(X) = \mathbb{Z}/2$, then the space form does not admit a free involution (see Section 2). For $X = \Sigma^{2n}$ and τ an arbitrary involution, $H^*(X/\tau; \mathbb{Z}/2)$ is isomorphic to the cohomology of the $2n$-dimensional projective space and by Lemma 5.1 below it follows that the Borsuk–Ulam theorem holds. So we will consider from now on only odd-dimensional homotopy spherical space forms.

2. Free involutions and equivariant maps

Definition 2.1. A free involution τ on a CW-complex X is a homeomorphism of X, which is fixed-point free and such that τ^2 is the identity map of X.

The existence of a free involution is equivalent to the existence of a free $\mathbb{Z}/2$ $(= \{1, -1\})$ action T, where $\tau(x) = T(x, -1)$. We will use the notation (X, τ) for a CW-complex X with a free involution τ.

Let (X, τ) be a CW-complex with a free involution. Consider the $\mathbb{Z}/2$-principal bundle $p : X \to X/\tau$. By classical bundle theory, p is classified by a class $[f]$ in $[X/\tau, \mathbb{RP}^\infty]$, or equivalently by a class $w_1(p)$ in $H^1(X/\tau; \mathbb{Z}/2)$, where the cohomology class $w_1(p) = f^* w_1(\gamma)$ is the pullback of the first Stiefel–Whitney class of the universal Hopf line bundle γ over $\mathbb{RP}^\infty$. Let E

be the m-dimensional vector bundle $X \times_{\mathbb{Z}/2} \mathbb{R}^m$ over $B = X/\tau$, where $\mathbb{Z}/2$ acts on X by τ and on $\mathbb{R}^m$ by the antipodal map -1.

Proposition 2.2. *Let (X, τ) be a CW-complex with a free involution. Then the following four conditions are equivalent:*

(1) *the Borsuk–Ulam theorem holds for the triple $(X, \tau; \mathbb{R}^m)$;*
(2) *there is not a $\mathbb{Z}/2$-equivariant map $f : X \to S^{m-1}$;*
(3) *there is not a map $f : X/\tau \to \mathbb{RP}^{m-1}$ such that the pullback of the nontrivial class of $H^1(\mathbb{RP}^{m-1}; \mathbb{Z}/2)$ is the first characteristic class of the $\mathbb{Z}/2$-bundle $X \to X/\tau$;*
(4) *every continuous cross section of the bundle $E \to B$ has a zero.*

Proof. Equivalence of (1) and (2) is clear, just consider the map $g(x) = \frac{f(x)-f(\tau(x))}{\|f(x)-f(\tau(x))\|}$ from X to S^{m-1}. To prove that (2) is equivalent to (3), consider the $\mathbb{Z}/2$-principal bundle $p : X \to X/\tau$, as above, and let p be classified by the class $[f]$ in $[X/\tau, \mathbb{RP}^\infty]$. There exists a $\mathbb{Z}/2$-equivariant map $\hat{f} : X \to S^\infty$ that covers f. Now assume condition (3) does not hold. This means that the map f factors through the $(m-1)$-skeleton of $\mathbb{RP}^{m-1}$, and then, by the universal property of the pullback, there exists a map $\hat{g} : X \to S^{m-1}$ that factors $\hat{f}$, and therefore is $\mathbb{Z}/2$-equivariant. Conversely, again by contradiction, if there is a $\mathbb{Z}/2$-equivariant map $f : X \to S^{m-1}$, then the induced map in the quotients satisfies the condition that the pullback of the nontrivial class of $H^1(\mathbb{RP}^{m-1}; \mathbb{Z}/2)$ is the first characteristic class of the $\mathbb{Z}/2$-bundle $X \to X/\tau$. For the equivalence of (1) and (4), let us first assume that (1) holds. A map $f : X \to \mathbb{R}^3$ gives a section as $s_f([x]) = [x, f(x) - f(\tau(x))]$, and s_f has a zero at $[x] \in B$ if and only if $f(x) = f(\tau(x))$. This contradicts (1) and the implication follows. Conversely, suppose that (4) holds. If s is a section which has no zeros, the function $f_s : X \to \mathbb{R}^3$ defined by $s([x]) = [x, f_s(x)]$ has the property that $f_s(\tau(x)) = -f(x) \neq 0$ for all $x \in X$, which contradicts (4). So the result follows. □

3. Periodic groups and spherical space forms

We recall in this section some facts about periodic groups and their use in the classification of homotopy spherical space forms. Main references are [AM], [Sw], [Sw1], [T1], [T2], [T3], [T4], [TW], [Wo]. A finite group G is said to be *periodic* and of *period* $p > 0$ if $H^p(G; \mathbb{Z})$ is cyclic of order $|G|$ with a generator g such that $\smile g : H^k(G; \mathbb{Z}) \to H^{k+p}(G; \mathbb{Z})$ is an isomorphism for all $k \geq 1$ (see [CE, p. 261]). Here $H^k(G; \mathbb{Z}) = H^k(BG; \mathbb{Z})$, where BG is the classifying space of G. The list of all finite periodic groups is given in Table 1, where $t \geq 3$, $i \geq 1$, and r, q are prime numbers, $r \geq 3$, $q \geq 5$. The groups Q_{2^t} (the presentation we are using is $\langle x, y : x^{2^{t-1}} = y^2, yxy^{-1} = x^{-1} \rangle$) are the generalized quaternionic groups and the groups T_i and O_i^*, the generalized binary tetrahedral and the generalized binary octahedral, are defined in [AM, p. 147]. We are using the notation $H \rtimes_\phi G$ for the semidirect product of G

Table 1. Periodic groups.

Family	Definition	Conditions
I	$\mathbb{Z}/a \rtimes_\phi \mathbb{Z}/b$	$(a, b) = 1$
II	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Q_{2^t})$	$(a, b) = (ab, 2) = 1$
III	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times T_i)$	$(a, b) = (ab, 6) = 1$
IV	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times O_i^*)$	$(a, b) = (ab, 6) = 1$
V	$(\mathbb{Z}/a \rtimes_\phi \mathbb{Z}/b) \times Sl_2(\mathbb{F}_r)$	$(a, b) = (ab, r(r^2 - 1)) = 1$
VI	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Tl_2(\mathbb{F}_q))$	$(a, b) = (ab, q(q^2 - 1)) = 1$

and H with respect to the action $\phi : G \to \mathrm{Aut}(H)$, so $H \rtimes_\phi G$ is the set $H \times G$ with the product defined by $(h, g)(h', g') = (h\phi(g)h', gg')$.

Let Σ^m denote an m-dimensional CW-complex with the homotopy type of the m-sphere. Given a free cellular action γ of a finite group G on Σ^m, let us denote by $\Sigma^m/\gamma(G)$ the corresponding orbit space, which is an m-dimensional homotopy spherical space form. This is also known as an *m-Swan complex*, or a *homotopy m-spherical space form* (see [Sw]). It is clear that $\pi_1(\Sigma^m/\gamma(G)) = G$. It is a well-known fact that finite groups acting freely on finite-dimensional homotopy sphere are periodic (see [CE, p. 355]). The opposite implication was proved by Swan, namely if a finite group G has period p, then G acts freely and cellularly on Σ^{p-1} [Sw, Proposition 4.1] (in particular, a group G acts freely and orientation preserving on a sphere S^{m-1} only if G is periodic of period m).

Definition 3.1. Let G be a finite group and $m \geq 4$. A (G, m)-polarization of a finite-dimensional complex X consists of an isomorphism $\pi_1(X) = G$ and a homotopy equivalence of the universal cover $\tilde{X} \simeq S^{m-1}$. Two polarized spaces X_1 and X_2 are equivalent if there exists a homotopy equivalence $X_1 \simeq X_2$ which preserves the polarization.

The main result in this context is the following (see [Sw], [TW, Definition 2.1, Theorems 2.2 and 2.7]).

Theorem 3.2. *If G is a finite group with period p, then*

(1) *the equivalence classes of (G, p)-polarized complexes correspond bijectively to the equivalent classes of the generators of $H^p(G; \mathbb{Z})$, where two generators a and b are equivalents if there is a $\varphi \in \mathrm{Aut}(G)$ such that $\varphi^*(a) = \pm b$;*
(2) *there exist $m > p$ and a finite complex with a (G, m)-polarization.*

The bijection stated in Theorem 3.2(2) is defined by associating with each (G, m)-polarized complex X its first nontrivial k-invariant, $k_{m+1}(X) \in H^{m+1}(X_{m-1}; \pi_m(X))$, with X_{m-1} the $(m-1)$-stage in the Postnikov tower of X.

TABLE 2. Periodic groups of period 2 or 4.

Family	Definition	Conditions
I	$\mathbb{Z}/c \times (\mathbb{Z}/a \rtimes_\phi \mathbb{Z}/2^t)$	$\phi(1) = -1.$ and $(a, 2^t) = (a, c) = (2^t, c) = 1$
II	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Q_{2^t})$	$\phi(x), \phi(y) \in \{\pm 1.\}$ and $(a, b) = (ab, 2) = 1$
III	$\mathbb{Z}/b \times T_i$	$(b, 6) = 1$
IV	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times O_i^*)$	$\phi(r) \in \{\pm 1.\}$ and $(a, b) = (ab, 6) = 1$
V	$\mathbb{Z}/a \times Sl_2(\mathbb{F}_5)$	$(a, 120) = 1$

In this paper, we will give special attention to the three-dimensional (homotopy) spherical space forms, and hence we need the groups which have period 2 or 4. From [T3] we have that the groups which have period 2 or 4 are those listed in Table 2, where the notation is the same as that of Table 1, and $\pm 1\cdot$ is defined as follows. Identify the group $\mathrm{Aut}(\mathbb{Z}/a)$ with the unit group $(\mathbb{Z}/a)^\times$ of the ring of the integral residue classes mod a. Then, $+1\cdot$ is the identity automorphism and $-1\cdot$ is the multiplication by -1.

4. Spherical space forms with free involutions

When does a given space form admit a free involution? We have the following first geometric characterization of this problem which is a direct consequence of basic results in covering space theory.

Proposition 4.1. *If a homotopy spherical space form $\Sigma^{m-1}/\eta(H)$ has a free involution, then H is a subgroup of index 2 of some other group G appearing in Table 1. Conversely, if H is a subgroup of index 2 of some other group G of Table 1, then there exists a homotopy spherical space form with fundamental group H having a free involution.*

Now we compare the space forms which arise from a group G and a subgroup H of index 2.

Proposition 4.2. *Let G and H be two groups appearing in Table 1 of period $2n$, and assume H is a subgroup of G of index 2, with inclusion $i : H \hookrightarrow G$. Then, the induced homomorphism $(Bi)^{2n} : H^{2n}(BG; \mathbb{Z}) \to H^{2n}(BH; \mathbb{Z})$ is surjective.*

Proof. The inclusion $i : H \hookrightarrow G$ induces a homomorphism

$$(i)^{2n} : H^{2n}(G; \mathbb{Z}) \to H^{2n}(H; \mathbb{Z}).$$

We have the transfer homomorphism $tr_H^G : H^{2n}(H; \mathbb{Z}) \to H^{2n}(G; \mathbb{Z})$, and the composite $tr_H^G(i)^{2n}$ is a multiplication by 2, by [Br, Proposition 9.5(ii)]. Since $H^{2n}(G; \mathbb{Z})$ and $H^{2n}(H; \mathbb{Z})$ are cyclic groups of order $|G|$ and $|H|$, respectively, and $|G| = 2 \cdot |H|$, the first homomorphism has to be surjective. □

TABLE 3. Subgroups of index 2 of the periodic groups.

Family	Type of group	Subgroups of index 2
I	$\mathbb{Z}/a \rtimes_\phi \mathbb{Z}/b$, $2a'b' = ab$	$\mathbb{Z}/a' \rtimes_\phi \mathbb{Z}/b'$
IIa	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Q_8)$	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times \mathbb{Z}/4)$
IIb	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Q_{2^t})$, $t > 3$	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times \mathbb{Z}_{2^{t-1}})$, $\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Q_{2^{t-1}})$
IV	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times O_i^*)$	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times T_i)$
VI	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Tl_2(\mathbb{F}_q))$	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Sl_2(\mathbb{F}_q))$

Proposition 4.3. *Let $\Sigma^{2n-1}/\eta(H)$ be a given $(2n-1)$-dimensional homotopy space form. A necessary and sufficient condition for the existence of a representative with a free involution in the equivalence class of $(H, 2n-1)$-polarized spaces of $\Sigma^{2n-1}/\eta(H)$ is that the group H is a subgroup of index 2 of some group G appearing in the list of either Table 1 (if $n = 2$) or Table 2 (if $n > 2$).*

Proof. Clearly sufficiency follows from Proposition 4.1. In order to prove necessity, let $i : H \hookrightarrow G$ denote the inclusion. By Proposition 4.2, the homomorphism

$$(Bi)^* : H^{2n-1}(BG; \mathbb{Z}) \to H^{2n-1}(BH; \mathbb{Z})$$

is onto. The two groups are cyclic and the homomorphism sends the set of generators onto the set of generators. Using Theorem 3.2 and the bijection claimed there, the above map between generators corresponds to space level taking a double covering and the result follows. □

Beside providing a proof of Theorem 1.1, Propositions 4.1 and 4.3 indicate that in order to know the free involutions on a given $(2n-1)$-dimensional space form, we should answer the following question: for a given group H in Table 1 (or Table 2 in case the period is 2 or 4), find all groups G in that table such that H is isomorphic to a subgroup of G of index 2. Namely, we have to identify the groups in Tables 1 and 2 that admit subgroups of index 2. By direct analysis of those tables (see also [T4]) we get Tables 3 and 4 where we list in the second column all the groups in the corresponding families of Table 1 (resp., Table 2) that admit subgroups of index 2, and in the third column we list, for each group, the subgroups of index 2 (up to isomorphism). The conditions defining the groups are those given in Table 1 (resp., Table 2), plus possible further conditions given here explicitly. The groups of Table 1 (resp., Table 2) not appearing in the second column of Table 3 (resp., Table 4) do not admit subgroups of index 2.

An immediate consequence of the results presented in these tables is the following.

Corollary 4.4. *A three-dimensional homotopy spherical space form with fundamental group H admits an involution if and only if H is one of the*

Table 4. Subgroups of index 2 of the periodic groups of period 2 or 4.

Family	Type of group	Subgroups of index 2
I	$\mathbb{Z}/c \times (\mathbb{Z}/a \rtimes_\phi \mathbb{Z}/2^t)$	$\mathbb{Z}/c \times (\mathbb{Z}/a \times \mathbb{Z}/2^{t-1})$
IIa	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Q_8)$	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times \mathbb{Z}/4)$
IIb	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Q_{2^t})$, $t > 3$	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times \mathbb{Z}_{2^{t-1}})$, $\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Q_{2^{t-1}})$
IV	$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times O_i^*)$	$\mathbb{Z}/a \times (\mathbb{Z}/b \times T_i)$

Table 5. Sylow 2-subgroups of the groups in Table 4.

Family	G_2	H_2
I	$\mathbb{Z}/2^t$ $(t > 1)$	$\mathbb{Z}/2^{t-1}$
IIa	Q_8	$\mathbb{Z}/4$
IIb	Q_{2^t} $(t > 3)$	$\mathbb{Z}_{2^{t-1}}$ and $Q_{2^{t-1}}$
IV	Q_{16}	Q_8

following groups: $\mathbb{Z}/a \times \mathbb{Z}/2^t$, $\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times \mathbb{Z}_4)$, $\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times \mathbb{Z}_{2^{t-1}})$ *for* $t > 3$, $\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Q_{2^{t-1}})$ *for* $t > 3$, $\mathbb{Z}/a \times (\mathbb{Z}/b \times T_i)$. *In the case of a* $(2n-1)$*-dimensional homotopy spherical space form* $(n > 2)$ *the groups* H *are the ones above plus the groups of the form* $\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Sl_2(\mathbb{F}_q))$.

Remark 4.5. Given an odd integer $2n-1$, the list of all finite groups which act freely in some $(2n-1)$-dimensional homotopy sphere can be a proper subset of Table 1. Nevertheless the groups $\mathbb{Z}/c \times (\mathbb{Z}/a \rtimes_\phi \mathbb{Z}/2^t)$ belong to the list for any odd integer $2n-1$.

We conclude this section with the list of the Sylow 2-subgroups of the groups in Table 4, necessary in the next section. Let G denote one of the group in the second column of Table 4, and H the associated subgroup in the third column. The Sylow 2-subgroups of these groups are denoted by G_2 and H_2 and are listed in Table 5. The result follows recalling that a and b are odd in all types except type I, that $T_i = Q_8 \rtimes_\alpha Z/3^i$, and that there are extensions (see [AM, p. 147])

$$1 \to T_i \to O_i^* \to \mathbb{Z}/2 \to 1.$$

5. The Borsuk–Ulam theorem

In this section we prove our main results. For a pair $(X(2n-1), \tau)$ where $X(2n-1)$ is a $(2n-1)$-dimensional homotopy spherical space form and τ is a free involution on $X(2n-1)$, we have a double covering $X(2n-1) \to B$

where $B = X(2n-1)/\tau$. Then $\pi_1(X(2n-1)) = H$ and $\pi_1(B) = G$, where the groups H and G are in Table 3, and we have the inclusion $H \hookrightarrow G$ defined by τ or by the double covering map $X(2n-1) \to B$. A pair of groups (H, G) means the group G and a subgroup H, where the subgroup H has index 2 in G. The set of such pairs is in one-to-one correspondence with the set of the pairs (G, θ), where $\theta : G \to \mathbb{Z}/2$ is an epimorphism. Recall that the subset of the nontrivial elements of $\mathrm{Hom}(G, \mathbb{Z}/2)$ is in one-to-one correspondence with the set of the nontrivial elements w_1 of $H^1(G; \mathbb{Z}/2)$. We will use this latter correspondence to obtain a preliminary formulation of the Borsuk–Ulam theorem. The next lemma relates our problem with a problem about the cohomology ring of the space B.

Lemma 5.1. *Given a pair $(X(2n-1), \tau)$ as above, consider the corresponding pair (G, w_1). Then, the Borsuk–Ulam theorem holds for the triple $(X(2n-1), \tau; \mathbb{R}^{2n-1})$ if and only if the cohomology class $(w_1)^{2n-1} \neq 0$.*

Proof. By Proposition 2.2 the Borsuk–Ulam theorem holds for the triple $(X(2n-1), \tau; \mathbb{R}^{2n-1})$ if and only if every continuous cross section of $E \to B$ has a zero. Obstruction theory says that the vector bundle $E \to B$ has a cross section which has no zeros if and only if the Euler class $e(E) \in H^{2n-1}(B; \mathbb{Z}^w)$ with twisted integers coefficients associated with the first Stiefel–Whitney class of E is zero. But $E = (2n-1)L$ (Whitney sum) where L is the line bundle $B \times_{\mathbb{Z}/2} \mathbb{R}$. So $e(E) = e(L)^{2n-1}$, which reduces mod 2 to w_1^{2n-1}. But reduction mod 2 : $H^{2n-1}(B; \mathbb{Z}^w) \to H^{2n-1}(B; \mathbb{Z}_2) = \mathbb{Z}_2$ is an isomorphism, because it is Poincaré dual to $H_0(B; \mathbb{Z}^w) \to H_0(B; \mathbb{Z}/2)$, since B is an orientable Poincaré complex and the local coefficient $\mathbb{Z}^w$ is nontrivial. Then the result follows. □

The next lemma summarizes known results about the cohomology of the periodic groups (see, for example, [AM]).

Lemma 5.2.

(1) *$H^*(\mathbb{Z}/a \rtimes_\phi \mathbb{Z}/2^t; \mathbb{Z}/2)$ is $\mathbb{Z}/2[u_1]$, for $t = 1$, and $\Lambda(u_1) \otimes \mathbb{Z}/2[v_2]$ for $t > 1$;*
(2) *$H^*(\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Q_{2^t}); \mathbb{Z}/2)$ is the algebra $\mathbb{Z}/2[e_4](1, x, y, x^2, y^2, x^2y = xy^2)$ and $x^3 = y^3 = 0$ for $t = 3$ and $\mathbb{Z}/2[x, y, e_4]/(xy = 0, x^3 = y^3)$ for $t \geq 4$;*
(3) *$H^*(\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times O_i^*); \mathbb{Z}/2)$ is the algebra $\mathbb{Z}/2[e_4](1, e, e^2, e^3)$ and $e^4 = 0$;*
(4) *$\mathbb{Z}/a \rtimes_\phi (\mathbb{Z}/b \times Tl_2(\mathbb{F}_q))$ is the algebra $\mathbb{Z}/2[e_4](1, e, e^2, e^3)$ and $e^4 = 0$.*

We are now in the position of proving Theorems 1.2 and 1.4. The corollaries follow by investigation of Tables 3 and 4. Consider first the three-dimensional case. Let H be a subgroup of index 2 in a group G acting on a homotopy sphere Σ^3, let $X = \Sigma^3/H$, $B = \Sigma^3/G$. Let G_2 be a Sylow 2-subgroup of G, so that $H_2 = H \cap G_2$ is a Sylow 2-subgroup of H. Consider the finite cover $B_2 = \Sigma^3/G_2 \to B$. Then, $w_1^3 \in H^3(B; \mathbb{Z}/2)$ is zero if and only if its lift is zero in $H^3(B_2; \mathbb{Z}/2)$. This follows because the composition

$$H^3(B; \mathbb{Z}/2) \to H^3(B_2; \mathbb{Z}/2) \to H^3(B; \mathbb{Z}/2)$$

of the pullback and the transfer is multiplication by $[G : G_2]$ (see [AM, Chapter II, Lemma 5.1]), which is 1 mod 2. Since the Sylow 2-subgroups of our groups are either cyclic $\mathbb{Z}/2^t$ ($t \geq 0$) or generalized quaternion Q_{2^t} ($t \geq 3$) by [AM, Chapter IV, Corollary 6.6], the next lemma follows promptly by Lemmas 5.1 and 5.2, and this completes the proof of Theorem 1.2.

Lemma 5.3. *Given pairs* $(X(3), \tau)$ *and* (G, w_1) *as in Lemma* 5.1, *then* w_1^3 *is*

(i) *nonzero if either* $G_2 = \mathbb{Z}/2$, $H_2 = 0$, *or* $G_2 = Q_{2^t}$, $H_2 = Q_{2^{t-1}}$, $t > 3$;

(ii) *zero if either* $G_2 = \mathbb{Z}/2^t$, $H_2 = \mathbb{Z}/2^{t-1}$, $t > 1$, *or* $G_2 = Q_{2^t}$, $H_2 = \mathbb{Z}/2^{t-1}$, $t \geq 3$.

Consider next the proof of Theorem 1.4. In this case the result follows by the very structure of the cohomology ring of the groups appearing as a fundamental group of the space Σ^{2n-1}/H, as can be see by inspection of Lemma 5.2.

Remark 5.4. The Borsuk–Ulam theorem always hold for maps from three-dimensional spherical space forms into $\mathbb{R}^2$, namely for the triple $(X, \tau; \mathbb{R}^2)$. In order to obtain this result it suffices to show that the surjective homomorphism $\theta : \pi_1(X/\tau) \to \mathbb{Z}/2$ never factors through $\mathbb{Z} \to \mathbb{Z}/2$. Since $\pi_1(X/\tau)$ is a finite group, $\mathrm{Hom}(\pi_1(X/\tau), \mathbb{Z})$ is trivial. Since θ is nontrivial, the result follows.

Remark 5.5. About the Borsuk–Ulam theorem for an m-dimensional CW-complex X and maps into $\mathbb{R}^k$ where $k < m$, some works have been done both in general and on specific spaces; see, for example, [DL]. Concerning space forms, this problem has been solved by Stolz in [St], when the space X is the projective space $\mathbb{RP}^{2n-1}$.

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